On the Integrated Form of the BBGKY Hierarchy for Hard Spheres

Herbert Spohn

Theoretische Physik, Ludwig-Maximilians-Universität,
Theresienstraße 37, 80333 München, Germany

March 1985

A note on the history (May 2006): In my book "Large Scale Dynamics of Interacting Particles" [S] I refer to an unpublished note from early 1985 on the BBGKY hierarchy for hard spheres. My main point there was to provide a direct probabilistic proof for the time-integrated version of the hierarchy. Over recent years there has been repeated interest in this derivation, which encourages me to make my note public. I decided to leave it in its original form including likely inaccuracies. The work of R. Illner and M. Pulvirenti [IP] appeared in September 1985, see also the book by C. Cercignani, R. Illner, and M. Pulvirenti [CIP], who prove the same result using special flow representation and methods from the theory of differential operators.

[S] H. Spohn, Large Scale Dynamics of Interacting Particles, Texts and Monographs in Physics, Springer-Verlag, Heidelberg, 1991.

[IP] R. Illner and M. Pulvirenti, A derivation of the BBGKY-hierarchy for hard sphere particle systems, Transport Theory and Stat. Phys. **16**, 997–1012 (1987), preprint DM-388-IR, September 1985.

[CIP] C. Cercignani, R. Illner, and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Applied Mathematical Sciences **106**, Springer-Verlag, New York, 1994.

The aim of these notes is to establish the integrated form of the BBGKY hierarchy for hard spheres as used by O.E. Lanford [1] in his proof of the validity of the Boltzmann equation in the Boltzmann–Grad limit. The idea of a direct probabilistic proof is inspired by a paper of R. Lang and X.X. Nguyen [2].

We denote by $x_j = (q_j, p_j) \in \Lambda \times \mathbb{R}^3$ position and momentum of the j–th particle. The hard spheres have diameter a (and mass one). They are confined to the region Λ. Λ is bounded and has a "smooth" boundary $\partial\Lambda$. Conditions on $\partial\Lambda$ ensuring the existence of the hard sphere dynamics are given in the thesis of K. Alexander [3], p. 13/14, and we assume the validity of these conditions here. We have exactly N particles, $j = 1, \dots, N$. The n-particle phase space, $n = 1, 2, \dots, N$, is

$$\begin{aligned}\Gamma_n = \big\{ &(x_1, \dots, x_n) \in (\Lambda \times \mathbb{R}^3)^n \big| \; |q_i - q| \geq a/2 \text{ for all} \\ &q \in \partial\Lambda, |q_i - q_j| \geq a \;, i, j = 1, \dots, n, i \neq j \big\} \,. \end{aligned} \tag{1}$$

In a collision of two hard spheres incoming and outgoing momenta transform into each other as

$$
\begin{aligned}
p_i' &= p_i - \hat\omega[\hat\omega\cdot(p_i - p_j)]\,,\\
p_j' &= p_j + \hat\omega[\hat\omega\cdot(p_i - p_j)]\,,
\end{aligned}
\tag{2}
$$

$i \neq j$, with $\hat\omega \in S^2$. At the wall particles are specularly reflected,

$$
p_j' = p_j - 2\hat n(q_j)[\hat n(q_j)\cdot p_j]\,, \tag{3}
$$

where $\hat n(q_j)$ is the unit outward normal at the point of contact. For the construction of the hard sphere dynamics we refer to Alexander′s thesis.

We remove once and for all from Γ_n the set of points which in the course of time run into either a grazing or a multiple collision. The phase space with these points removed is denoted by Γ_n^*. $\Gamma_n \setminus \Gamma_n^*$ has Lebesgue measure zero. Then, for all $t \in \mathbb{R}$ and for every point $(x_1,\dots,x_n) \in \Gamma_n^*$, the flow

$$
t \mapsto T_t^{(n)}(x_1,\dots,x_n) \in \Gamma_n^* \tag{4}
$$

is well defined. In particular Γ_n^* is invariant under $T_t^{(n)}$.

If incoming and outgoing momenta are identified (also at collisions with the wall), then $T_t^{(n)}$ is continuous in t, i.e. for all $(x_1,\dots,x_n) \in \Gamma_n^*$ one has

$$
\lim_{t\to 0}\; T_t^{(n)}(x_1,\dots,x_n) = (x_1,\dots,x_n)\,. \tag{5}
$$

Here we will *not* identify incoming and outgoing momenta, i.e. we regard them as distinct phase points. For $(x_1,\dots,x_n) \in \Gamma_n^*$ the map $t \mapsto T_t^{(n)}(x_1,\dots,x_n)$ is then piecewise continuous and we have to distinguish between the limit from the future $(+)$ and from the past $(-)$ defined by

$$
T_{t\pm}^{(n)}(x_1,\dots,x_n) = \lim_{\varepsilon\to 0,\varepsilon>0}\; T_{t\pm\varepsilon}^{(n)}(x_1,\dots,x_n)\,. \tag{6}
$$

If the added signs $\pm$ are omitted, it is understood that the quantity in question is independent of the way the limit is taken.

A function $\rho_n : \Gamma_n \to \mathbb{R}$ is continuous along trajectories of $T_t^{(n)}$ on Γ_n^*, if for all $(x_1,\dots,x_n) \in \Gamma_n^*$

$$
\lim_{t\to 0}\; \rho_n(T_t^{(n)}(x_1,\dots,x_n)) = \rho_n(x_1,\dots,x_n)\,, \tag{7}
$$

where both the limit from the future and the past are understood. This implies then that for all $(x_1,\dots,q_i,p_i,\dots,q_i+a\hat\omega,p_j,\dots,x_n) \in \Gamma_n^*$ one has

$$
\begin{aligned}
&\rho_n(x_1,\dots,q_i,p_i,\dots,q_i+a\hat\omega,p_j,\dots,x_n)\\
&= \rho_n(x_1,\dots,q_i,p_i',\dots,q_i+a\hat\omega,p_j',\dots,x_n)
\end{aligned}
\tag{8}
$$

and similarly for collisions with the wall.

For future convenience we define some sets: Let

$$\Gamma_{N-n}(x_1,\ldots,x_n) = \Big\{(x_{n+1},\ldots,x_N)\in\Gamma_{N-n}\Big|\ |q_i-q_j|\geq a, \\ \text{for } i=1,\ldots,n \text{ and } j=n+1,\ldots,N\Big\} \tag{9}$$

for $(x_1,\ldots,x_n)\in\Gamma_n^*$ and

$$\Gamma_{N-n}(x_1,\ldots,x_n)=\emptyset \tag{10}$$

otherwise. Let

$$\Omega_j(x_1,\ldots,x_n,p_{n+1}) = \Big\{\hat\omega\in S^2\Big|(x_1,\ldots,x_n,q_j+a\hat\omega,p_{n+1})\in\Gamma_{n+1}^*\Big\}\subset S^2 \tag{11}$$

for $j=1,\ldots,n$, $(x_1,\ldots,x_n)\in\Gamma_n^*$, and $p_{n+1}\in\mathbb{R}^3$ and let

$$\Omega_j(x_1,\ldots,x_n,p_{n+1})=\emptyset \tag{12}$$

otherwise. We define two subsets, $\Omega_{j\pm}$, of Ω_j by

$$\Omega_{j\pm}(x_1,\ldots,x_n,p_{n+1}) = \Big\{\hat\omega\in\Omega_j(x_1,\ldots,x_n,p_{n+1})\Big|\hat\omega\cdot(p_{n+1}-p_j)>0\ (<0)\Big\}. \tag{13}$$

After these preparations we can state our assumptions on the initial ($t=0$) measure.

Let P be the initial probability measure on Γ_N. P is assumed to satisfy:
(i) P is symmetric in the particle labels.
(ii) P has a density,

$$P(dx_1\ldots dx_N)=f_N(x_1,\ldots,x_N)dx_1\ldots dx_N. \tag{14}$$

(iii) f_N is bounded by the canonical equilibrium distribution, i.e. there exist constants $c,\beta>0$ such that

$$f_N(x_1,\ldots,x_N)\leq c\prod_{j=1}^N h_\beta(p_j) \tag{15}$$

on Γ_N, where $h_\beta(p)=(\frac{\beta}{2\pi})^{3/2}e^{-\beta p^2/2}$ is the normalized Maxwellian.
(iv) $f_N=0$ on $\Gamma_N\setminus\Gamma_N^*$ and f_N is continuous along trajectories of $T_t^{(N)}$ on Γ_N^*, cf. (7) and (8).
(v) The time evolved measure P_t has a density $f_N(t)$ given by $f_N(t)=0$ on $\Gamma_N\setminus\Gamma_N^*$ and

$$f_N(x_1,\ldots,x_N,t)=f_N(T_{-t}^{(N)}(x_1,\ldots,x_N)) \tag{16}$$

for $(x_1,\ldots,x_N)\in\Gamma_N^*$. The canonical equilibrium measure is denoted by P_{eq}.

To avoid confusion we remark that identities are always understood pointwise. If they hold only a.s., we state this explicitly. Often we will work with densities of

measures. As in the case of $f_N(t)$ and of $\rho_n(t)$ below we will choose then a specific version.

Because of a definite number of particles, N, the correlation functions are, up to a multiplicative factor, just the marginal measures. We fix a particular version of these measures by

$$\rho_n(x_1,\dots,x_n,t) = N\dots(N-n+1)\int_{\Gamma_{N-n}(x_1,\dots,x_n)} dx_{n+1}\dots dx_n f_N(x_1,\dots,x_N,t)\,. \tag{17}$$

Note that $\rho_n(t)=0$ on $\Gamma_n\setminus\Gamma_n^*$ by our definition of $\Gamma_{N-n}(x_1,\dots,x_n)$. Let $\Delta\subset\Gamma_n$ be a Borel set. We remove a set of Lebesgue measure zero to guarantee that $\Delta\subset\Gamma_n^*$. Then because of the hard core exclusion and by symmetry

$$\begin{aligned}&\int_\Delta dx_1\dots dx_n\rho_n(x_1,\dots,x_n,t)\\ &\qquad = N\dots(N-n+1)P\{(x_1(t),\dots,x_n(t))\in\Delta\}\,,\end{aligned} \tag{18}$$

where for $j=1,\dots,N$ we set

$$x_j(t\pm,x) = (T^{(N)}_{t\pm}\,x)_j\,. \tag{19}$$

The probability in (18) is independent of whether the limit is taken from the future of from the past.

To avoid an overburdened language it is convenient to set

$$t>0\,,$$

which we do from now on. This is no restriction, of course.

Let $\tau_m\ge 0$, $m=1,2,\dots$, be the time of the m–th collision between the set of particles with labels $1,\dots,n$ and the set of particles with labels $n+1,\dots,N$. If there are simultaneous collisions between the two groups of particles, then they are ordered according to the label in the first group.

Proposition 1 *The following idendity holds for all Borel sets $\Delta\subset\Gamma_n^*$, for all $n=1,2,\dots,N$,*

$$\begin{aligned}&\int_\Delta dx_1\dots dx_n\rho_n(x_1,\dots,x_n,t)\\ &\quad=\int_\Delta dx_1\dots dx_n\rho_n(T^{(n)}_{-t}(x_1,\dots,x_n))\\ &\qquad+\sum_{m=1}^\infty N\dots(N-n+1)P\big\{(x_1(\tau_m+),\dots,x_n(\tau_m+))\in T^{(n)}_{\tau_m-t+}\Delta,\tau_m\le t\big\}\\ &\qquad-\sum_{m=1}^\infty N\dots(N-n+1)P\big\{(x_1(\tau_m-),\dots,x_n(\tau_m-))\in T^{(n)}_{\tau_m-t-}\Delta,\tau_m\le t\big\}\,.\end{aligned} \tag{20}$$

Proof: We use inclusion - exclusion to obtain

$$\begin{aligned}
&P\big\{(x_1(t),\dots,x_n(t))\in\Delta\big\}\\
&\quad = P\big\{(x_1(t-),\dots,x_n(t-))\in\Delta,\tau_1\le t\big\}\\
&\qquad +P\big\{(x_1(t+),\dots,x_n(t+))\in\Delta,\tau_1> t\big\}\\
&\quad = \sum_{m=1}^{\infty} P\big\{(x_1(t-),\dots,x_n(t-))\in\Delta,\tau_{m+1}>t,\tau_m\le t\big\}\\
&\qquad +P\big\{(x_1,\dots,x_n)\in T^{(n)}_{-t-}\Delta,\tau_1> t\big\}\\
&\quad = \sum_{m=1}^{\infty} P\big\{(x_1(\tau_m+),\dots,x_n(\tau_m+))\in T^{(n)}_{\tau_m-t+}\Delta,\tau_{m+1}>t,\tau_m\le t\big\}\\
&\qquad +P\big\{(x_1,\dots,x_n)\in T^{(n)}_{-t}\Delta\big\}-P\big\{(x_1,\dots,x_n)\in T^{(n)}_{-t-}\Delta,\tau_1\le t\big\}\\
&\quad = \sum_{m=1}^{\infty} P\big\{(x_1(\tau_m+),\dots,x_n(\tau_m+))\in T^{(n)}_{\tau_{m-t+}}\Delta,\tau_m\le t\big\}\\
&\qquad -\sum_{m=1}^{\infty} P\big\{(x_1(\tau_m+),\dots,x_n(\tau_m+))\in T^{(n)}_{\tau_{m-t+}}\Delta,\tau_{m+1}\le t\big\}\quad (*)\\
&\qquad +P\big\{(x_1,\dots,x_n)\in T^{(n)}_{-t}\Delta\big\}\\
&\qquad -\sum_{m=1}^{\infty} P\big\{(x_1(\tau_m-),\dots,x_n(\tau_m-))\in T^{(n)}_{\tau_{m-t-}}\Delta,\tau_m\le t\big\}\\
&\qquad +\sum_{m=1}^{\infty} P\big\{(x_1(\tau_{m+1}-),\dots,x_n(\tau_{m+1}-))\in T^{(n)}_{\tau_{m+1}-t-}\Delta,\tau_{m+1}\le t\big\}\,.\quad (**)
\end{aligned}\tag{21}$$

To justify (21) we need an integrable bound. Clearly, the m-th term is bounded by $cP_{\mathrm{eq}}\{\tau_m\le t\}$. We will show in Lemma 2 below that this bound is summable.

In (21) $(*)$ and $(**)$ cancel each other because as sets

$$\begin{aligned}
&\big\{(x_1,\dots,x_N)\in\Gamma_N^*\big|(x_1(\tau_m+),\dots,x_n(\tau_m+))\in T^{(n)}_{\tau_m-t+}\Delta,\tau_{m+1}\le t\big\}\\
&=\big\{(x_1,\dots,x_N)\in\Gamma_N^*\big|(x_1(\tau_{m+1}-),\dots,x_n(\tau_{m+1}-))\in T^{(n)}_{\tau_{m+1}-t-}\Delta,\tau_{m+1}\le t\big\}\,.
\end{aligned}\tag{22}$$

□

Lemma 2 *Let $\tau_m\ge 0$, $m=1,2,\dots$, be the time of the m-th collision for the system of N hard spheres (collisions with the wall are not counted). Then*

$$\begin{aligned}
&\sum_{m=1}^{\infty} P_{\mathrm{eq}}\{\tau_m\le t\}\\
&\quad = t\int dq_1dp_1\int dp_2\int_{\Omega_{1-}(x_1,p_2)} d\hat\omega\ a^2\hat\omega\cdot(p_1-p_2)\rho_{\mathrm{eq},2}(q_1,p_1,q_1+a\hat\omega,p_2)\,.
\end{aligned}\tag{23}$$

Proof: We think of the hard sphere dynamics as a flow under a function (special flow), cf. [4] for this construction in our context, and we prove Lemma 2 first for this case.

Let B be the base and $T : B \to B$ be an invertible map which preserves the finite measure μ. Let $h : B \to \mathbb{R}_+$ be the ceiling function. We assume that h is integrable. The phase space is then $\Gamma = \{x \in B, y \in \mathbb{R}_+ \,|\, 0 \le y \le h(x)\}$. The flow T_t is constructed piecewise in the following way: $T_t : (x, y) \mapsto (x, y + t)$ until the first time for which $y + t = h(x)$. Then $(x, h(x)) \mapsto (Tx, 0)$. We refer to this transformation as a collision. The construction is then continued into the future and the past. The measure $\mu(dx) \times dy = P_{\mathrm{eq}}$ is invariant under T_t. Let $\tau_m \ge 0$, $m = 1, 2, \ldots$, be the time of the m-th collision. Then we claim that

$$\sum_{m=1}^{\infty} P_{\mathrm{eq}}\{\tau_m \le t\} = t\mu(B)\,. \tag{24}$$

Since $h > 0$, $\sum_{j=0}^{\infty} h(T^{-j}x) = \infty$ $\mu(dx)$ a.s. by the Poincaré recurrence theorem. Therefore

$$B_k = \big\{x \in B \big| \sum_{j=0}^{k-2} h(T^{-j}x) \le t, \sum_{j=0}^{k-1} h(T^{-j}x) > t\big\}, \tag{25}$$

$k = 1, 2, \ldots$, forms a partition of B. Then

$$\begin{aligned}\sum_{m=1}^{\infty} P\{\tau_m \le t\} &= t\mu(B_1) + \sum_{k=2}^{\infty} \int_{B_k} \mu(dx) h(x) \\ &\quad + \sum_{m=2}^{\infty} \Big\{ \int_{B_m} \mu(dx)\big(t - \sum_{j=0}^{m-2} h(T^{-j}x)\big) + \sum_{k=m+1}^{\infty} \int_{B_k} \mu(dx) h(T^{-k+2}x)\Big\} \\ &= t\sum_{k=1}^{\infty} \mu(B_k) = t\mu(B)\,. \end{aligned} \tag{26}$$

For hard spheres the base consists of configurations with outgoing momenta and is defined by

$$\begin{aligned} B = \big\{(x_1, \ldots, x_N) \in \Gamma_N^* \big| &\text{ there exists a pair } (i, j), i \ne j, \\ &\text{such that} \quad q_j = q_i + a\hat{\omega}, (p_j - p_i) \cdot \hat{\omega} > 0\big\}. \end{aligned} \tag{27}$$

The ceiling function is defined as the time until the next collision (not counting collisions with the wall). The equilibrium measure induces on B the invariant surface measure

$$\Big\{ \sum_{i \ne j = 1}^{N} a^2 dq_i d\hat{\omega}\ \hat{\omega} \cdot (p_j - p_i) \prod_{k=1, k \ne i,j}^{N} dq_k \Big\} \prod_{k=1}^{N} h_\beta(p_k) dp_k\,. \tag{28}$$

Its total weight is given by (23). □

We want to express (20) in terms of correlation functions. For this purpose we first have to show some regularity of these functions.

Lemma 3 *Under our assumptions on P, for every $s \in \mathbb{R}$, $\rho_n(s) = 0$ on $\Gamma_n \setminus \Gamma_n^*$ and $\rho_n(s)$ is continuous along trajectories of $T_t^{(n)}$ on Γ_n^*.*

Proof: Since, by assumptions (iv) and (v), $f_N(s)$ has the same continuity properties as $f_N(0)$, we may set $s = 0$.

To simplify notation we abbreviate $x = (x_1, \ldots, x_n), y = (x_{n+1}, \ldots, x_N)$ and we set $x(t\pm, x) = T_{t\pm}^{(n)} x$. For $x \in \Gamma_n^*$ let $\Lambda(x, t) \subset \Lambda$ be spatial region traced out by the particles' motion $x(s), 0 \leq s \leq t$, with initial condition x. We set $\Lambda(x, 0) = \Lambda(x)$. Correspondingly we define $\Lambda(y, t) \subset \Lambda$ for $y \in \Gamma_{N-n}^*$. Then for $x \in \Gamma_n^*$ let

$$\Gamma_{N-n}(x, t) = \big\{ y \in \Gamma_{N-n}^* | \Lambda(x, t) \cap \Lambda(y, t) = \emptyset \big\}. \tag{29}$$

We have $\Gamma_{N-n}(x) = \Gamma_{N-n}(x, 0)$ up to a set of dy-measure zero. For $x \in \Gamma_n^*$ we define the flow $T_t^{(x)}$ on $\Gamma_{N-n}(x)^*$ of $N - n$ particles in the spatial region $\Lambda \setminus \Lambda(x)$. Here the * indicates again that we remove from $\Gamma_{N-n}(x)$ a set of Lebesgue measure zero on which the flow remains undefined.

With these definitions, for $x \in \Gamma_n^*$,

$$\begin{aligned} \rho_n(T_{t\pm}^{(n)} x) &= \int_{\Gamma_{N-n}(x(t))} dy N \ldots (N - n + 1) f_N(T_{t\pm}^{(n)} x, y) \\ &= \int_{\Gamma_{N-n}(x(t))} dy N \ldots (N - n + 1) f_N(T_{t\pm}^{(n)} x, T_t^{(x(t))} y) \end{aligned} \tag{30}$$

by Liouville's theorem for the map $T_t^{(x(t))}$ for fixed t.

We choose now a τ such that $0 < t \leq \tau$. For $x \in \Gamma_n^*$ and $y \in \Gamma_{N-n}(x, \tau)$ the "x"-particles and the "y"-particles do not interact during the time interval $[0, \tau]$. We then have two possibilities:

(1) The time evolution exists into the future and the past, i.e. $(x, y) \in \Gamma_N^*$. In this case $(T_{t\pm}^{(n)} x, T_{t\pm}^{(x(t))} y) = T_{t\pm}^{(N)}(x, y)$. We denote the set of such y's by $\widehat{\Gamma}_{N-n}(x, \tau)$.

(2) The time evolution does not exist, i.e. $(x, y) \notin \Gamma_N^*$. In this case, by assumption (v),

$$f_N(T_t^{(n)} x, T_t^{(x(t))} y) = 0 \quad \text{for} \quad 0 \leq t \leq \tau. \tag{31}$$

Therefore, for every $x \in \Gamma_n^*$,

$$\begin{aligned} &|\rho_n(T_{t\pm}^{(n)} x) - \rho_n(x)| \\ &\quad = N \ldots (N - n + 1) \Big| \int_{\widehat{\Gamma}_{N-n}(x,\tau)} dy f_N(T_t^{(N)}(x, y)) \\ &\quad + \int_{\Gamma_{N-n}(x(\tau)) \setminus \Gamma_{N-n}(x,\tau)} dy f_N(T_{t\pm}^{(n)} x, T_t^{(x(t))} y) - \int_{\Gamma_{N-n}(x)} dy f_N(x, y) \Big| \\ &\quad \leq N \ldots (N - n + 1) \Big\{ \int_{\widehat{\Gamma}_{N-n}(x,\tau)} dy |f_N(T_t^{(n)}(x, y)) - f_N(x, y)| \\ &\quad + c \int_{\Gamma_{N-n}(x(\tau)) \setminus \Gamma_{N-n}(x,\tau)} dy f_{\mathrm{eq},N}(x, y) + c \int_{\Gamma_{N-n}(x) \setminus \Gamma_{N-n}(x,\tau)} dy f_{\mathrm{eq},N}(x, y) \Big\}. \end{aligned} \tag{32}$$

The last two terms are bounded by *const.* τ. For fixed τ the first term vanishes in the limit $t \to 0$. This follows from dominated convergence and our assumption (iv). $\square$

Lemma 4 *Under our assumptions on P, for every $(x_1, \dots, x_n) \in \Gamma_n^*$ the map $t \mapsto \rho_n(x_1, \dots, x_n, t)$ is continuous, i.e.*

$$\lim_{t \to 0} \rho_n(x_1, \dots, x_n, s+t) = \rho_n(x_1, \dots, x_n, s) \,. \tag{33}$$

Proof: We use the same notation as in the proof of Lemma 3. Since, by assumptions (iv) and (v), $f_N(s)$ has the same continuity properties as $f_N(0)$, we may set $s = 0$.

For every $x \in \Gamma_n^*$ we have

$$\begin{aligned} &\rho_n(x,t) - \rho_n(x) \\ &\quad = \int_{\{y|(x,y)\in\Gamma_N^*\}} dy f_N(x,y,t) - \int_{\{y|(x,y)\in\Gamma_N^*\}} dy f_N(x,y) \\ &\quad = \int_{\{y|(x,y)\in\Gamma_N^*\}} dy (f_N(T_{-t}^{(N)}(x,y)) - f_N(x,y)) \,. \end{aligned} \tag{34}$$

The claim follows then by dominated convergence from assumption (iv). $\square$

Proposition 5 *The following identity holds for every Borel set $\Delta \subset \Gamma_n^*$, $n = 1, 2, \dots, N$,*

$$\begin{aligned} &\int_\Delta dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) \\ &\quad = \int_\Delta dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\ &\qquad + \sum_{j=1}^n \int_0^t ds \int_\Delta dx_1 \dots dx_n \, [C_{j,n+1}\rho_{n+1}(s)] \, (T_{-t+s}^{(n)}(x_1, \dots, x_n)) \,. \end{aligned} \tag{35}$$

Here the collision operator is defined by

$$\begin{aligned} &(C_{j,n+1}\rho_{n+1}(s))(x_1, \dots, x_n) \\ &\quad = a^2 \int dp_{n+1} \int_{\Omega(x_1,\dots,x_n,p_{n+1})} d\hat\omega \hat\omega \cdot (p_{n+1} - p_j) \rho_{n+1}(x_1, \dots, x_n, q_j + a\hat\omega, p_{n+1}, s) \,. \end{aligned} \tag{36}$$

Proof: We consider the third term of (20), cf. Proposition 1. For $0 \le s < t$ we want to compute the limit as $\varepsilon \to 0$ of

$$\begin{aligned} &\sum_{m=1}^\infty \frac{1}{\varepsilon} P\big\{(x_1(\tau_m-), \dots, x_n(\tau_m-)) \in T_{\tau_m - t-}^{(n)} \Delta, \tau_m \in [s, s+\varepsilon]\big\} \\ &\quad = \sum_{k=1}^\infty k \frac{1}{\varepsilon} P_s \big\{\text{particles with labels } 1, \dots, n \text{ collide exactly } k \text{ times with} \\ &\qquad\qquad \text{particles with labels } n+1, \dots, N \text{ during the time interval } [0, \varepsilon], \\ &\qquad\qquad \text{at the times } \tau \text{ of collision } (x_1(\tau-), \dots, x_n(\tau-)) \in T_{s+\tau-t-}^{(n)} \Delta\big\} \,. \end{aligned} \tag{37}$$

Here P_s is the measure P evolved to time s. By assumption (iii) the sum for $k \geq 2$ is bounded by

$$\sum_{k=2}^{\infty} k \frac{c}{\varepsilon} P_{\text{eq}} \big\{ \text{particles have exactly } k \text{ collisions during the time interval } [0, \varepsilon] \big\}. \quad (38)$$

By the same argument as in Lemma 2

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P_{\text{eq}} \big\{ \text{particles have exactly one collision during the time interval } [0, \varepsilon] \big\} \\ &\quad = \int dq_1 dp_1 \int dp_2 \int_{\Omega_{1-}(x_1, p_2)} d\hat{\omega}\ a^2 \hat{\omega} \cdot (p_1 - p_2) \rho_{\text{eq},2}(q_1, p_1, q_1 + a\hat{\omega}, p_2) \end{aligned} \quad (39)$$

and according to Lemma 2

$$\begin{aligned} &\sum_{k=1}^{\infty} k P_{\text{eq}} \big\{ \text{particles have exactly } k \text{ collisions during the time interval } [0, \varepsilon] \big\} \\ &\quad = \varepsilon \int dq_1 dp_1 \int dp_2 \int_{\Omega_{1-}(x_1, p_2)} d\hat{\omega}\ a^2 \hat{\omega} \cdot (p_1 - p_2) \rho_{\text{eq},2}(q_1, p_1, q_1 + a\hat{\omega}, p_2)\,. \end{aligned} \quad (40)$$

Therefore in the limit $\varepsilon \to 0$ the expression in (38) vanishes.

We are left with the term $k = 1$ of (37). Let us label the particle at the collision with $n + 1$. Then we have to compute the limit $\varepsilon \to 0$ of

$$\begin{aligned} &(N - n) \frac{1}{\varepsilon} P_s \big\{ \text{particles with label } 1, \dots, n \text{ collide exactly once with particle } n + 1 \\ &\qquad \text{and do not collide with particles with label } n + 2, \dots, N \text{ during the time} \\ &\qquad \text{interval } [0, \varepsilon], \text{ at the time } \tau \text{ of collision } (x_1(\tau-), \dots, x_n(\tau-)) \in T^{(n)}_{s+\tau-t-} \Delta \big\} \\ &= \sum_{j=1}^{n} (N - n) \frac{1}{\varepsilon} P_s \big\{ \text{ the only collision during time interval } [0, \varepsilon] \text{ is} \\ &\qquad \text{between particle } j \text{ and particle } n + 1, \text{ at time } \tau \text{ of collision} \\ &\qquad (x_1(\tau-), \dots, x_n(\tau-)) \in T^{(n)}_{s+\tau-t-} \Delta \big\} + \mathcal{O}(\varepsilon)\,. \end{aligned} \quad (41)$$

The error is bounded by

$$\frac{c}{\varepsilon} P_{\text{eq}} \big\{ \text{there is more than one collision during the time interval } [0, \varepsilon] \big\}\,, \quad (42)$$

which vanishes in the limit $\varepsilon \to 0$.

Let

$$\begin{aligned} A_j(\varepsilon) = \big\{ (x_1, \dots, x_{n+1}) \in \Gamma^*_{n+1} \big| T^{(n+1)}_t (x_1, \dots, x_{n+1}) \text{ for } 0 \leq t \leq \varepsilon \\ \text{has as only collision the one between particles } j \text{ and } n + 1, \\ \text{at time } \tau \text{ of collision } (x_1(\tau-), \dots, x_n(\tau-)) \in T^{(n)}_{s+\tau-t-} \Delta \big\}. \end{aligned} \quad (43)$$

In the definition (43) $x_j(t\pm,x)=(T^{(n+1)}_{t\pm}x)_j$. If the set defined in (41) is called $B_j(\varepsilon)$, then

$$\begin{aligned}
(41) &= \sum_{j=1}^{n}(N-n)\frac{1}{\varepsilon}\int_{B_j(\varepsilon)} dx_1\dots dx_N f_N(x_1,\dots,x_N,s)\\
&= \sum_{j=1}^{n}(N-n)\frac{1}{\varepsilon}\int_{A_j(\varepsilon)} dx_1\dots dx_{n+1}\int_{\Gamma_{N-n-2}(x_1,\dots,x_{n+1})} dx_{n+2}\dots dx_N f_N(x_1,\dots,x_N,s)\\
&\quad+\sum_{j=1}^{n}(N-n)\frac{1}{\varepsilon}\Big[\int_{B_j(\varepsilon)} dx_1\dots dx_N f_N(x_1,\dots,x_N,s)\\
&\quad-\int_{A_j(\varepsilon)} dx_1\dots dx_{n+1}\int_{\Gamma_{N-n-2}(x_1,\dots,x_{n+1})} dx_{n+2}\dots dx_N f_N(x_1,\dots,x_N,s)\Big]\,. \qquad (44)
\end{aligned}$$

The second term is again bounded by (42) and vanishes therefore in the limit $\varepsilon\to 0$.

Multiplying with the factor $N\dots(N-n+1)$ of (20) we are left with

$$\sum_{j=1}^{n}\frac{1}{\varepsilon}\int_{A_j(\varepsilon)} dx_1\dots dx_{n+1}\rho_{n+1}(x_1,\dots,x_{n+1},s)\,. \qquad (45)$$

Let τ, $0\le\tau\le\varepsilon$, be the time of collision. Then on $A_j(\varepsilon)$

$$q_{n+1}=q_j+a\hat\omega+\tau(p_j-p_{n+1})\,. \qquad (46)$$

We perform this substitution in the integral (45). The change in volume element is

$$dq_jdp_jdq_{n+1}dp_{n+1}=a^2\hat\omega\cdot(p_j-p_{n+1})dq_jdp_jd\tau d\hat\omega dp_{n+1}\,. \qquad (47)$$

We flow on $A_j(\varepsilon)$ all $n+1$ coordinates from time 0 to time τ. Then

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{1}{\varepsilon}\int_{A_j(\varepsilon)} dx_1\dots dx_{n+1}\rho_{n+1}(x_1,\dots,x_{n+1},s)\\
&\quad=\sum_{j=1}^{n}\frac{1}{\varepsilon}\int_0^\varepsilon d\tau a^2\int_{T^{(n)}_{s+\tau-t-}\Delta} dx_1\dots dx_n\int dp_{n+1}\int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega\hat\omega\cdot(p_j-p_{n+1})\\
&\qquad\quad\times\chi_\Xi(x_1,\dots,x_n,\hat\omega,p_{n+1})\rho_{n+1}(T^{(n+1)}_{-\tau}(x_1,\dots,x_n,q_j+a\hat\omega,p_{n+1}),s)\\
&\quad=\sum_{j=1}^{n}\frac{1}{\varepsilon}\int_0^\varepsilon d\tau a^2\int_{T^{(n)}_{s+\tau-t-}\Delta} dx_1\dots dx_n\int dp_{n+1}\int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega\hat\omega\cdot(p_j-p_{n+1})\\
&\qquad\quad\times\rho_{n+1}(T^{(n+1)}_{-\tau}(x_1,\dots,x_n,q_j+a\hat\omega,p_{n+1}),s)+\mathcal{O}(\varepsilon)\,. \qquad (48)
\end{aligned}$$

In the second integral χ_Ξ is the indicator function of the set $\{x_1,\dots,x_n,\hat\omega,p_{n+1}|$ $T^{(n+1)}_{\tau'}(x_1,\dots,x_n,q_j+a\hat\omega,p_{n+1})$ for $-\tau\le\tau'\le\varepsilon-\tau$ has only one collision$\}$. [As collisions we always refer to collisions between two particles and not to collisions

with the wall. Therefore in (46) we should actually use the free flow of particles j and $n+1$ separately *including* collisions with the wall. After flowing the $n+1$ coordinates to time τ we still obtain (48).] The error term is again bounded by (42). Only $P_{\rm eq}$ refers now to the equilibrium measure of $n+1$ particles.

To obtain the limit as $\varepsilon \to 0$ of (48) we have to show that the integrand is continuous at $\tau = 0$. To see this we bound as

$$
\begin{aligned}
&\Big| \int_{T^{(n)}_{s+\tau-t}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega \hat\omega \cdot (p_j - p_{n+1}) \\
&\qquad\qquad \times \rho_{n+1}(T^{(n+1)}_{-\tau}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}), s) \\
&\quad - \int_{T^{(n)}_{s-t}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega \hat\omega \cdot (p_j - p_{n+1}) \\
&\qquad\qquad \times \rho_{n+1}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}, s) \Big| \\
&\le \int_{(T^{(n)}_{s+\tau-t}\Delta \cup T^{(n)}_{s-t}\Delta)\setminus(T^{(n)}_{s+\tau-t}\Delta \cap T^{(n)}_{s-t}\Delta)} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega \\
&\qquad\qquad \times \hat\omega \cdot (p_j - p_{n+1}) \rho_{n+1}(T^{(n+1)}_{-\tau}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}), s) \\
&\quad + \int_{T^{(n)}_{s-t}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega \hat\omega \cdot (p_j - p_{n+1}) \\
&\times \big| \rho_{n+1}(T^{(n+1)}_{-\tau}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}), s) - \rho_{n+1}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}, s) \big| \,.
\end{aligned}
\tag{49}
$$

In the first term we bound $\rho^{(s)}_{n+1}$ by $const. f_{{\rm eq},n+1}$. By dominated convergence this term vanishes then in the limit $\tau \to 0$. In the second term we integrate only over points such that $(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}) \in \Gamma^*_{n+1}$. Therefore by Lemma 3 the integrand is continuous in τ and vanishes as $\tau \to 0$.

Altogether we have shown that the measure

$$
\sum_{m=1}^{\infty} N \dots (N-n+1) P\{(x_1(\tau_m-),\dots,x_n(\tau_m-)) \in T^{(n)}_{\tau_m - t-}\Delta, \tau_m \in ds\}
\tag{50}
$$

is absolutely continuous with respect to the Lebesgue measure and has a density given by

$$
\begin{aligned}
\sum_{j=1}^{n} a^2 \int_{T^{(n)}_{s-t}\Delta} dx_1 \dots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\dots,x_n,p_{n+1})} d\hat\omega \hat\omega \cdot (p_j - p_{n+1}) \\
\times \rho_{n+1}(x_1,\dots,x_n,q_j + a\hat\omega, p_{n+1}, s)\,.
\end{aligned}
\tag{51}
$$

We note that by Lemma 4 this density is continuous in s.

The same argument applied to the second term of (20) shows that

$$
\sum_{m=1}^{\infty} N \dots (N-n+1) P\{(x_1(\tau_m+),\dots,x_n(\tau_m+)) \in T^{(n)}_{\tau_m - t+}\Delta, \tau_m \in ds\}
\tag{52}
$$

has a density given by

$$\sum_{j=1}^{n} a^2 \int_{\{(x_1,\ldots,q_j,p_j',\ldots,x_n)\in T_{s-t}^{(n)}\Delta\}} dx_1\ldots dx_n \int dp_{n+1} \int_{\Omega_{j-}(x_1,\ldots,x_n,p_{n+1})} d\hat{\omega}\hat{\omega}\cdot(p_j-p_{n+1})$$
$$\times \rho_{n+1}(x_1,\ldots,x_n,q_j+a\hat{\omega},p_{n+1},s)$$
$$= \sum_{j=1}^{n} a^2 \int_{\{(x_1,\ldots,q_j,p_j',\ldots,x_n)\in T_{s-t}^{(n)}\Delta\}} dx_1\ldots dq_j dp_j'\ldots dx_n \int dp_{n+1}' \tag{53}$$
$$\times \int_{\Omega_{j+}(x_1,\ldots,q_j,p_j',\ldots,x_n,p_{n+1}')} d\hat{\omega}\hat{\omega}\cdot(p_{n+1}'-p_j')\rho_{n+1}(x_1,\ldots,q_j,p_j',\ldots,x_n,q_j+a\hat{\omega},p_{n+1}',s),$$

where we used again Lemma 3 which ensures that on the domain of integration $\rho_{n+1}^{(s)}$ is continuous through a collision. We relabel in (53) (p_j', p_{n+1}') as (p_j, p_{n+1}) and subtract (51) from (53). Since $\Delta\subset\Gamma_n^*$,

$$\sum_{j=1}^{n}\int_0^t ds \int_{T_{s-t}^{(n)}\Delta} dx_1\ldots dx_n [C_{j,n+1}\rho_{n+1}(s)](x_1,\ldots,x_n)$$
$$= \sum_{j=1}^{n}\int_0^t ds \int_{\Delta} dx_1\ldots dx_n [C_{j,n+1}\rho_{n+1}(s)](T_{s-t}^{(n)}(x_1,\ldots,x_n))\,. \tag{54}$$

□

To obtain the integrated from of the BBGKY hierarchy we have to iterate (35). For this purpose we go back to (45). Since we integrate there over a Borel set of Γ_{n+1}, we could have chosen any other version of $\rho_{n+1}(s)$, i.e. any other function $\tilde{\rho}_{n+1}(s)$ such that $\rho_{n+1}(s)=\tilde{\rho}_{n+1}(s)\,dx_1\ldots dx_{n+1}$ a.s.. We used however certain properties of $\rho_{n+1}(s)$ in the proof below (45). Therefore, if we want to replace $\rho_{n+1}(s)$ by $\tilde{\rho}_{n+1}(s)$, the latter has to satisfy:

(1) $\tilde{\rho}_{n+1}(s)=\rho_{n+1}(s)$ a.s..

(2) For fixed s, $\tilde{\rho}_{n+1}(s)$ is continuous along trajectories of $T_t^{(n+1)}$ on Γ_{n+1}^*.

(3) For every $(x_1,\ldots,x_{n+1})\in\Gamma_{n+1}^*$, $s\mapsto\tilde{\rho}_{n+1}(x_1,\ldots,x_{n+1},s)$ is continuous.

(4) There exist constants c',β such that

$$\tilde{\rho}_{n+1}(s)\le c' f_{\mathrm{eq},n+1}^{(\beta)}\,. \tag{55}$$

Corollary 6 *Let $\tilde{\rho}_{n+1}(s) : \Gamma_{n+1} \to \mathbb{R}$ satisfy the Properties (1) to (4) given above. Then*

$$\begin{aligned}
&\int_\Delta dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) \\
&= \int_\Delta dx_1 \dots dx_n \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\
&\quad + \sum_{j=0}^{n} \int_0^t ds \int_\Delta dx_1 \dots dx_n [C_{j,n+1}\tilde{\rho}_{n+1}(s)](T_{-t+s}^{(n)}(x_1, \dots, x_n)) \, . \qquad (56)
\end{aligned}$$

Lemma 7 *Let $\hat{\rho}_n(t)$ be defined by*

$$\begin{aligned}
\hat{\rho}_n(x_1, \dots, x_n, t) &= \rho_n(T_{-t}^{(n)}(x_1, \dots, x_n)) \\
&\quad + \sum_{j=1}^{n} \int_0^t ds [C_{j,n+1}\tilde{\rho}_{n+1}(s)](T_{-t+s+}^{(n)}(x_1, \dots, x_n)) \qquad (57)
\end{aligned}$$

for every point $(x_1, \dots, x_n) \in \Gamma_n^*$, where $\tilde{\rho}_{n+1}(s)$ satisfies the above Properties (1) to (4). Then $\tilde{\rho}_n(t)$ satisfies also the above four properties.

We note that $\hat{\rho}_n(t) = 0$ on $\Gamma_n \setminus \Gamma_n^*$ by the definition of ρ_n and of $C_{j,n+1}$.

Proof: Property (1) follows from Corollary 6. Property (3) follows from Lemma 3 (continuity of ρ_n along trajectories of $T_t^{(n)}$ on Γ_n^*) and from Property (4) of $\tilde{\rho}_{n+1}(s)$.

For Property (4) we note that by Assumption (iii) on the initial measure

$$\rho_n(t) \le c' f_{\mathrm{eq},n}^{(\beta)} \, . \qquad (58)$$

Together with the assumed bound on $\tilde{\rho}_{n+1}(s)$ this gives a bound of the desired form with some new constants c'', β'.

For Property (2) the first term of (57) is continuous along trajectories by Lemma 3. Therefore we only have to consider the second term. We have

$$\begin{aligned}
&\Big| \sum_{j=1}^{n} \int_0^t ds [C_{j,n+1}\tilde{\rho}_{n+1}(s)](T_{-t+s+\tau+}^{(n)}(x_1, \dots, x_n)) \\
&\quad - \sum_{j=1}^{n} \int_0^t ds [C_{j,n+1}\tilde{\rho}_{n+1}(s)](T_{-t+s+}^{(n)}(x_1, \dots, x_n)) \Big| \\
&\le \sum_{j=1}^{n} \Big| \{ \int_0^\tau ds + \int_t^{t+\tau} ds \} | [C_{j,n+1}\tilde{\rho}_{n+1}(s-\tau)](T_{-t+s+}^{(n)}(x_1, \dots, x_n)) | \Big| \\
&\quad + \sum_{j=1}^{n} \int_0^t ds \Big| [C_{j,n+1}\tilde{\rho}_{n+1}(s-\tau) - C_{j,n+1}\tilde{\rho}_{n+1}(s)](T_{-t+s+}(x_1, \dots, x_n)) \Big| \, . \qquad (59)
\end{aligned}$$

By assumption the integrand of the first term is bounded by $c'' f^{(\beta')}_{\mathrm{eq},n}$. This implies that the first term vanishes in the limit $\tau \to 0$. In the second term we use dominated convergence. By the assumed continuity of $s \mapsto \tilde{\rho}_{n+1}(x_1, \ldots, x_{n+1}, s)$ for $(x_1, \ldots, x_{n+1}) \in \Gamma^*_{n+1}$ the integrand vanishes pointwise in the limit $\tau \to 0$. $\square$

By Lemma 7 we may set in (56) $\tilde{\rho}_{n+1}(s) = \hat{\rho}_{n+1}(s)$ and obtain

$$\int_\Delta dx_1 \ldots dx_n \rho_n(x_1, \ldots, x_n, t) = \int_\Delta dx_1 \ldots dx_n \Big[\rho_n(T^{(n)}_{-t+}(x_1, \ldots, x_n)) \tag{60}$$
$$+ \sum_{j_1=1}^{n} \int_0^t dt_1 (C_{j_1,n+1}(\rho_{n+1} \circ T^{(n+1)}_{-t_1+}))(T^{(n)}_{-t+t_1+}(x_1, \ldots, x_n))$$
$$+ \sum_{j_1=1}^{n} \sum_{j_2=1}^{n+1} \int_0^t dt_1 \int_0^{t_1} dt_2 (C_{j_1,n+1}(C_{j_2,n+2}\tilde{\rho}_{n+2}(t_2)) \circ T^{(n+1)}_{t_2-t_1+}))(T^{(n)}_{t_1-t+}(x_1, \ldots, x_n)) \Big] .$$

We iterate $N - n$ times and obtain

Proposition 8 *The following identity holds for every Borel set $\Delta \subset \Gamma^*_n$, $n = 1, \ldots, N$,*

$$\int_\Delta dx_1 \ldots dx_n \rho_n(x_1, \ldots, x_n, t) = \tag{61}$$
$$\sum_{m=0}^{N-n} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n+m-1} \int_0^t dt_1 \ldots \int_0^{t_{m-1}} dt_m \int_\Delta dx_1 \ldots dx_n$$
$$\times (C_{j_1,n+1} \ldots (C_{j_m,n+m}(\rho_{n+m} \circ T^{(n+m)}_{t_m+})) \circ T^{(n+m-1)}_{t_m - t_{m-1}+} \ldots)(T^{(n)}_{t_1-t+}(x_1, \ldots, x_n)) .$$

In a way Proposition 8 is our final result. There are however two reasons for reorganizing somewhat the integral (61). First of all we would like to get rid of the continuity assumptions, i.e. we would like to extend the validity of (61) to a more general class of initial measures. Secondly the Boltzmann–Grad limit is not quite apparent in the given form of (61).

We introduce the notion of a *collision history.* I choose this name to distinguish it from a sequence of real collisions of the N-particle system. One should remember that the correlation functions are averaged quantities. The correspondence between collision histories and sequences of real collisions is only very indirect.

A collision history is specified by the following list:

(a) $n \in \mathbb{N}$,

(b) $m \in \mathbb{N} \cup \{0\}$,

[If, as assumed so far, the number of particles equals N, then $1 \leq n \leq N$ and $0 \leq m \leq N - n$.]

(c) $(x_1, \ldots, x_n) \in \Gamma^*_n$,

(d) $(t_1, \dots, t_m) \in \mathbb{R}^m$

with the constraint $0 \le t_m \le \dots \le t_1 \le t$, $m \ge 1$,

(e) $(j_1, \dots, j_m) \in \mathbb{N}^m$

with the constraint $1 \le j_1 \le n, \dots, 1 \le j_m \le n+m-1$, $m \ge 1$,

(f) $(\hat{p}_1, \dots, \hat{p}_m) \in \mathbb{R}^{3m}$, $m \ge 1$,

(g) $(\hat{\omega}_1, \dots, \hat{\omega}_m) \in (S^2)^m$, $m \ge 1$,

with a complicated constraint depending on $n, m, x_1, \dots, x_n, t_1 \dots, t_m, j_1, \dots, j_m$, $\hat{p}_1, \dots, \hat{p}_m$ which is defined below. For future convenience we abbreviate a collision history as $(x_1, \dots, x_n, \delta)$, where δ stands for $(m, t_1, \dots, t_m, j_1, \dots, j_m, \hat{p}_1, \dots, \hat{p}_m$, $\hat{\omega}_1, \dots, \hat{\omega}_m)$.

Given the collision history $(x_1, \dots, x_n, \delta)$ we construct an evolution of particles in the following way. We choose n particles at $(x_1, \dots, x_n) \in \Gamma_n^*$. We consider this as the phase point at time t, $(x_1, \dots, x_n) \equiv (x_1(t), \dots, x_n(t))$ and we evolve backwards in time up to $t = 0$. We evolve the phase point $(x_1(t), \dots, x_n(t))$ to $T^{(n)}_{-t+t_1+}(x_1(t), \dots, x_n(t)) \equiv (x_1(t_1), \dots, x_n(t_1))$. We add a particle with label $n+1$ at $q_{j_1}(t_1) + a\hat{\omega}_1$ with momentum $\hat{p}_1$. We require that $\hat{\omega}_1 \in \Omega_{j_1}(x_1(t_1), \dots, x_n(t_1), \hat{p}_{n+1})$, i.e. $(x_1(t_1), \dots, x_n(t_1), q_{j_1}(t_1) + a\hat{\omega}, \hat{p}_1) \in \Gamma^*_{n+1}$. We call this new phase point of $n+1$ particles $(x_1(t_1), \dots, x_{n+1}(t_1))$ and evolve it to $T^{(n+1)}_{-t_1+t_2+}(x_1(t_1), \dots, x_{n+1}(t_1)) \equiv$ $(x_1(t_2), \dots, x_{n+1}(t_2))$. We add a particle with label $n+2$ at $q_{j_2}(t_2) + a\hat{\omega}_2$ with momentum $\hat{p}_2$. We require that $\hat{\omega}_2 \in \Omega_{j_2}(x_1(t_2), \dots, x_{n+1}(t_2), \hat{p}_{n+2})$. We call this new phase point of $n+2$ particles $(x_1(t_2), \dots, x_{n+2}(t_2))$. The final step is to evolve $(x_1(t_m), \dots, x_{n+m}(t_m)) \in \Gamma^*_{n+m}$ to
$T^{(n+m)}_{-t_m+}(x_1(t_m), \dots, x_{n+m}(t_m)) \equiv (x_1(0), \dots, x_{n+m}(0))$. If $m = 0$, then we only evolve $(x_1(t), \dots, x_n(t)) \equiv (x_1, \dots, x_n)$ to $T^{(n+m)}_{-t+}(x_1(t), \dots, x_n(t)) \equiv (x_1(0), \dots, x_n(0))$. To make the dependence on $x_1, \dots, x_n, \delta$ explicit we write $x_k(s, x_1, \dots, x_n, \delta)$ for $0 \le s \le t$, in particular $x_k(x_1, \dots, x_n, \delta) \equiv x_k(0)$, $k = 1, \dots, n+m$.

Let

$$\Delta(x_1, \dots, x_n; [0, t])$$

be the space of all collision histories for given n, starting configuration $(x_1, \dots, x_n) \in \Gamma_n^*$ and time span $[0, t]$. $\Delta(x_1, \dots, x_n; [0, t])$ is a subset of

$$\bigcup_{m \ge 0} \bigcup_{j_1=1}^{n} \dots \bigcup_{j_m=1}^{n+m-1} (\mathbb{R} \times \mathbb{R}^3 \times S^2)^m$$

defined by the above construction. We define a measure $d\delta$ on $\Delta(x_1, \dots, x_n;\ [0, t])$: $d\delta$ is the counting measure with respect to the discrete indices $m, j_1, \dots, j_m$ and the Lebesgue measure otherwise.

Given a collision history $(x_1, \dots, x_n, \delta)$ we define the weight function by

$$W(x_1, \dots, x_n, \delta) = \prod_{k=1}^{m} \{a^2 \hat{\omega}_k \cdot (\hat{p}_k - p_{j_k}(t_k, x_1, \dots, x_n, \delta))\} \,. \tag{62}$$

Lemma 9 *The following identity holds for every Borel set $\Delta \subset \Gamma_n^*$, $n \in \mathbb{N}$,*

$$\begin{aligned} \int_\Delta dx_1 \dots dx_n \rho_n(x_1, \dots, x_n, t) = \int_\Delta dx_1 \dots dx_n \int_{\Delta(x_1, \dots, x_n; [0,t])} d\delta \\ \times W(x_1, \dots, x_n, \delta) \rho_{n+m(\delta)}(x_1(x_1, \dots, x_n, \delta), \dots, x_{n+m(\delta)}(x_1, \dots, x_n, \delta)) \,. \end{aligned} \tag{63}$$

Proof: We write out (61) using the definition of $C_{j,n+1}$. From Assumption (iii) it follows that

$$\rho_{n+m} \le c'' f_{\mathrm{eq},n+m} \,. \tag{64}$$

(The simpler part of) Lanford's estimate on the uniform, in $a^2 N$, convergence of the BBGKY hierarchy shows that

$$\begin{aligned} &\int_\Delta dx_1 \dots dx_n \int_{\Delta(x, \dots, x_n; [0,t])} d\delta |W(x_1, \dots, x_n, \delta)| \\ &\qquad \times f^{(\beta)}_{\mathrm{eq},n+m(\delta)}(x_1(x_1, \dots, x_n, \delta), \dots, x_{n+m(\delta)}(x_1, \dots, x_n, \delta)) \\ &\quad \le c''' \int_\Delta dx_1 \dots dx_n f^{(\beta')}_{\mathrm{eq},n}(x_1, \dots, x_n) \end{aligned} \tag{65}$$

with $\beta' < \beta$ for all t. The details of this estimate can be found in F. King's thesis [5]. Therefore the integrations in (61) may be interchanged freely. □

In the form (63) we can extend our identity to a more general class of initial measures. In particular we will remove the restriction of a definite number of particles.

Let Γ be the grand canonical phase space,

$$\Gamma = \bigcup_{n \ge 0} \Gamma_n. \tag{66}$$

The grand canonical equilibrium measure with inverse temperature $\beta > 0$ and fugacity $z > 0$ is defined by

$$f^{(z,\beta)}_{\mathrm{eq},n}(x_1, \dots, x_n) \frac{1}{n!} dx_1 \dots dx_n = \frac{1}{Z} \prod_{j=1}^{n} \{z h_\beta(p_j)\} \frac{1}{n!} dx_1 \dots dx_n \tag{67}$$

on $\Gamma_n, n = 0, 1, \dots,$ where Z is the normalization constant. Let $\mathcal{C}$ be the class of functions $f : \Gamma \to \mathbb{R}$ such that

(i) f_n is measurable,

(ii) f_n is symmetric in the particle labels,

(iii) there exist positive constants M, z, β such that

$$|f_n(x_1, \ldots, x_n)| \leq M f_{\mathrm{eq},n}^{(z,\beta)}(x_1, \ldots, x_n) \tag{68}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $n = 0, 1, \ldots$.

Note that actually $\Gamma_n = \emptyset$ for sufficiently large n because of the hard core exclusion.

Given $f \in \mathcal{C}$ we define the "correlation function vector" $\rho : \Gamma \to \mathbb{R}$ by

$$\rho_n(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Gamma_m(x_1,\ldots,x_n)} dx_{n+1} \ldots dx_{n+m} f_{n+m}(x_1, \ldots, x_{n+m}) \tag{69}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n$, $n = 0, 1, \ldots$.

Lemma 10 *Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ be the map defined by (69). Then $\mathcal{F}$ is one–to–one and onto.*

Proof: We have by assumption (iii)

$$\begin{aligned} |\rho_n(x_1, \ldots, x_n)| &\leq M \sum_{m=0}^{\infty} \frac{1}{m!} \int dx_{n+1} \ldots dx_{n+m} f_{\mathrm{eq},n+m}^{(z,\beta)}(x_1, \ldots, x_{n+m}) \\ &\leq M e^{|\Lambda| z} f_{\mathrm{eq},n}^{(z,\beta)}(x_1, \ldots, x_n) \,. \end{aligned} \tag{70}$$

The inverse map is given by

$$f_n(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int dx_{n+1} \ldots dx_{n+m} \rho_{n+m}(x_1, \ldots, x_{n+m}) \,. \tag{71}$$

□

Let P be a signed measure on Γ with density f. Then the time evolved measure P_t has the density $f(t)$ with one version given by

$$f_n(x_1, \ldots, x_n, t) = f_n(T_{-t+}^{(n)}(x_1, \ldots, x_n)) \tag{72}$$

for all $(x_1, \ldots, x_n) \in \Gamma_n^*$ and $f_n(t) = 0$ on $\Gamma_n \setminus \Gamma_n^*$, $n = 1, 2, \ldots$. If $f \in \mathcal{C}$ then also $f(t) \in \mathcal{C}$ and the correlation functions at time t are still defined by (69).

Theorem 11 *Let P be a signed measure on Γ with density $f \in \mathcal{C}$. Then for every Borel set $\Delta \subset \Gamma_n$, $n = 1, 2, \ldots$, the following identity holds*

$$\begin{aligned} &\int_\Delta dx_1 \ldots dx_n \rho_n(x_1, \ldots, x_n, t) \\ &\quad = \int_\Delta dx_1 \ldots dx_n \int_{\Delta(x_1,\ldots,x_n;[0,t])} d\delta W(x_1, \ldots, x_n, \delta) \\ &\qquad \times \rho_{n+m(\delta)}(x_1(x_1, \ldots, x_n, \delta), \ldots, x_{n+m(\delta)}(x_1, \ldots, x_n, \delta)) \,. \end{aligned} \tag{73}$$

The right (left) hand side of (73) does not depend on the chosen version of ρ ($\rho(t)$).

Corollary 12 *Under the same assumptions, independently of the chosen versions of $\rho(t)$ and ρ,*

$$\rho_n(x_1,\ldots,x_n,t) = \int_{\Delta(x_1,\ldots,x_n,[0,t])} d\delta W(x_1,\ldots,x_n,\delta)$$
$$\times\rho_{n+m(\delta)}(x_1(x_1,\ldots,x_n,\delta),\ldots,x_{n+m(\delta)}(x_1,\ldots,x_n,\delta)) \tag{74}$$

$dx_1\ldots dx_n$ *a.s..*

Proof: Since the sum over m is finite, we consider a term with fixed m for reasons of notational simplicity. We abbreviate $x=(x_1,\ldots,x_n)$. Then (73) reads

$$\int_\Delta dx\rho_n(x,t) = \int_{\Delta^*} dx \int_{\Delta(x;[0,t],m)} d\delta W(x,\delta)\rho_{n+m}(y(x,\delta)) \tag{75}$$

with the obvious definition of $y(x,\delta)\in\Gamma^*_{n+m}$. Here we replaced Δ by $\Delta^*=\Delta\cap\Gamma^*_n$ which leaves the integral unchanged. Let $\mathcal{C}^*\subset\mathcal{C}$ be the class of densities f which satisfy the continuity assumptions (iv) and (v), cf. (16), omitting the requirements of normalization, positivity, and definite number of particles. If $f\in\mathcal{C}^*$, then (75) holds by Lemma 9. For an arbitrary $f\in\mathcal{C}$ there exists a sequence $f^\varepsilon\in\mathcal{C}$ such that

$$\lim_{\varepsilon\to0} f^\varepsilon = f \quad \text{a.s.} \tag{76}$$

on Γ. Consequently

$$\lim_{\varepsilon\to0}\rho^\varepsilon=\rho \tag{77}$$

and, since

$$\lim_{\varepsilon\to0} f^\varepsilon(t) = f(t) \quad \text{a.s.}\,, \tag{78}$$

we also have

$$\lim_{\varepsilon\to0}\rho^\varepsilon(t)=\rho(t) \quad \text{a.s.}\,. \tag{79}$$

Therefore

$$\lim_{\varepsilon\to0}\int_\Delta dx\rho^\varepsilon_n(x,t) = \int_\Delta dx\rho_n(x,t)\,. \tag{80}$$

We have to investigate now the convergence of the right hand side of (75). Let $\hat\rho\in\mathcal{C}^*$ and $\hat\rho=0$ a.s.. Then also $\hat\rho(t)=0$ a.s. and we conclude that

$$0=\int_{\Delta^*} dx\int_{\Delta(x;[0,t],m)} d\delta W(x,\delta)\hat\rho_{n+m}(y(x,\delta))\,. \tag{81}$$

Since all sets of measure zero can be approximated in this way, we conclude that y^{-1} (considered as a mapping for sets) maps sets in Γ^*_{n+m} of $dx_1\ldots dx_{n+m}$-measure zero to sets in $\{(x,\delta)|x\in\Delta^*,\delta\in\Delta(x;[0,t],m)\}$ of $dxd\delta$-measure zero.

By (77) there exists a set $\widehat{\Gamma} \subset \Gamma^*_{n+m}$ such that $\Gamma_{n+m} \setminus \widehat{\Gamma}$ has measure zero and such that $\lim_{\varepsilon \to 0} \rho^\varepsilon_{n+m} = \rho_{n+m}$ pointwise on $\widehat{\Gamma}$. Let $\chi_{\hat{\Gamma}}$ be the indicator function of the set $\widehat{\Gamma}$. Then

$$\lim_{\varepsilon \to 0} \rho^\varepsilon_{n+m} \chi_{\hat{\Gamma}}(y(x,\delta)) = \rho_{n+m} \chi_{\hat{\Gamma}}(y(x,\delta)) \tag{82}$$

and by the argument given above

$$\rho^\varepsilon_{n+m}(1 - \chi_{\hat{\Gamma}})(y(x,\delta)) = 0 \quad dx d\delta \quad \text{a.s.}\,. \tag{83}$$

Together with (65) our claim follows from dominated convergence. □

References

[1] O.E. Lanford, Time Evolution of Large Classical Systems. In: Dynamical Systems, Theory and Applications, edited by J. Moser. Lecture Notes in Physics **38**, p.1-111. Springer, Berlin, 1975.

[2] R. Lang and X.X. Nguyen, Smulochowski's theory of coagulation in colloids holds rigorously in the Boltzmann–Grad limit, Z. Wahrscheinlichkeitstheorie verw. Geb. **54**, 227-280 (1980).

[3] R.K. Alexander, The infinite hard sphere system, Ph.D. Thesis, Dep. of Mathematics, University of California at Berkeley, 1975.

[4] C. Marchicro, A. Pellegrinotti, E. Presutti and M. Pulvirenti, On the dynamics of particles in a bounded region: a measure theoretical approach, J. Math. Phys. **17**, 647-652 (1976).

[5] F. King, BBGKY hierarchy for positive potentials, Ph. D. Thesis, Dep. of Mathematics, University of California at Berkeley, 1975.